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Epicyclic Motion of Satellites About an Oblate Planet

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An analytic formulation is presented of a near circular orbit of a satellite about an axisymmetric potential. The model has a simple analytic form that is capable of describing all of the gravitational perturbative effects. Unlike more rigorous treatments, our approach has a simple geometric interpretation and greater mathematical simplicity than conventional descriptions of the perturbed motion, yet is sufficiently accurate to describe the motion of a satellite over its lifetime. The formulation of the orbit, based on epicyclic motion, is presented, and how all of the terms in the geopotential can be incorporated into the same formulation is demonstrated. Also considered are second-order effects arising from the J_2 term, which provides sufficient accuracy in the modeling for most satellite applications. Some simulation results are presented to establish the accuracy of the predictive model over time.

I. Introduction

THE effects of a nonspherical Earth on the orbits of satellites has been studied for around 40 years. Extensive analysis has been made showing the principal long-term effects that cause the orbits of satellites to evolve slowly in time. The standard approaches to this problem are based on expressing the evolution of the satellite orbits in terms of instantaneous Keplerian orbital elements or osculating elements. The evolution equations expressed in this form were attributed to Gauss in the study of perturbative effects in planetary motion. When the perturbing forces are conservative, as with gravitational perturbations, then the forces can be expressed in terms of the gradients of a disturbing function, expressed as a function of the osculating elements.

In 1959, Kozai[1] and Brouwer[2] produced analytic solutions of the perturbed orbital elements of a satellite. Kozai's[1] solutions included short-periodic variations up to first order in the perturbing forces and secular variations up to second order. In his paper, the disturbing potential was restricted to axially symmetric terms because these produce the more significant effects on the orbit. His solutions, however, had singularities for either small eccentricity or small inclination orbits. He later reformulated his solutions to overcome the problem for small eccentricity.[3]

Kozai[1] introduced the modified equinoctial coordinates $e(\cos\omega$, $e\sin\omega$, and the argument of latitude $M+\omega$, to replace e, ω, and $M)$ and derived the perturbation in terms of these orbital elements. In this way he was able to remove terms that had a small divisor of e. Kozai's approach was investigated further by Cook,[4] who showed that the zonal disturbing function can be written in a convenient manner by neglecting the short-periodic terms and expanding Kozai's solution in $e\cos\omega$ and $e\sin\omega$ up to an arbitrary odd zonal harmonic order.

For the even zonal harmonics, only the dominant J_2 effect was taken into account and terms of $\mathcal{O}(e^2)$ are ignored. Douglas and Ingram[5] expanded the periodic variations in the elements a, i, Ω, $e\cos\omega$, $e\sin\omega$, and $M+\omega$ up to $\mathcal{O}(J_2 e)$, whereas Izsak[6] developed the small eccentricity solutions to include $\mathcal{O}(J_2^2)$ terms. More recently, further investigations have been made by Gooding,[7] who developed complete, untruncated in eccentricity, first-order perturbation formulas via recurrence relations.

In this paper we propose a slightly different approach, based on the early work of King-Hele.[8] Our approach is to expand the coordinates of the satellite in an Earth-centered inertial frame, for small eccentricity. The motivation for this study is the realization that a large number of communication and remote sensing satellites are in orbits with eccentricities of order $\mathcal{O}(J_2)$ or smaller. This provides a natural ordering scheme to use for the perturbed equations. For these orbits the mathematical description of their complex motions can be greatly simplified compared to the full rigor of Brouwer's approach.[2] We start with the description of an epicycle orbit in three dimensions assuming a spherical Earth. We then show how these expressions can be slightly modified to produce the full perturbed motions and to derive the effects of each term in the Earth's potential. Like the Brouwer approach, our description of the location of the satellite does combine position and velocity terms. The position of the satellite is given by a redundant set of four coordinates, two to describe the instantaneous orbital plane and two which locate the position of the satellite on that plane. This provides a simple geometric interpretation of the description.

For satellites orbiting about the Earth, the J_2 potential is much larger than the rest of the zonal harmonics, and this means that second-order effects arising from J_2 are as important as the effects of J_3 and J_4. In this paper, we address the question of how the epicycle description can be extended to incorporate these second-order J_2 effects. In first-order theory it is immaterial whether one uses mean or osculating elements in the description, because differences are of order J_2, and the resulting errors will be of order J_2^2. In second-order theory, however, we cannot ignore such differences, and one must be careful when handling the first-order terms. Our description of an orbit uses osculating elements to define the orbital plane and polar coordinates to define position on the plane, and use is made of only one mean orbital element, the mean radius a. This is easily defined in terms of orbital energy, which is conserved for this problem.

In Sec. II we briefly review the description of epicycle orbits around a spherical Earth and interpret these basic descriptive equations. In Sec. III we derive the linearized equations of motion in the variables we have chosen and the fourth equation needed to solve for them. We provide analytic solutions to these equations for each term in the zonal harmonic expansion in Sec. IV and provide explicit descriptions of the motion for the first few terms in Appendix A.

In Sec. V, we begin by considering the postepicycle problem for Keplerian orbits. In Sec. VI, we develop a second-order theory for the J_2 term, before presenting comparisons of these analytic formulas with accurate numerical simulations of satellite orbits. The explicit descriptions of the $\mathcal{O}(J_2^2)$ coefficients are summarized in Appendix B.

In Sec. VII, we show comparisons of these analytic formulas with accurate numerical simulations of satellite orbits, and finally in Sec. VIII, we present a discussion of the work presented.

II. Epicycle Description of an Orbit

Keplerian Motion

We begin by describing briefly the epicycle motion of satellites in low-eccentricity orbits, about a spherical Earth. We start by describing the motion on the orbital plane and then develop our redundant

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representation of the full three-dimensional motion. The Keplerian equations of motion are

$$\ddot{r} - r\dot{\lambda}^2 = -\frac{\mu}{r^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\lambda}) = 0 \tag{1}$$

described in some polar coordinates (r, λ), where μ is the gravitational parameter. The azimuthal equation integrates to provide the angular momentum $h = r^2\dot{\lambda}$, and the radial equation integrates to provide the energy

$$\mathcal{E} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\lambda}^2) - \mu/r \tag{2}$$

For circular orbits we can find a solution to Eqs. (1) in which $r = a$ and $\dot{\lambda} = n$, where both a and n are constants that satisfy $a^3n^2 = \mu$. We are interested in the motions of satellites in very low-eccentricity orbits, which can be found by perturbing the preceding solution. For this, let $r = a + s$ and $\dot{\lambda} = n + \dot{\epsilon}$. Ignoring second-order terms in these small corrections, we can derive linear equations for them from Eqs. (1):

$$\frac{\ddot{s}}{an^2} + \frac{s}{a} = 2\left(\frac{2s}{a} + \frac{\dot{\epsilon}}{n}\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2s}{a} + \frac{\dot{\epsilon}}{n}\right) = 0 \tag{3}$$

Integrating these equations, we find

$$s = 2a\delta - A\cos(M - M_0)$$

$$\epsilon = \lambda_0 - 3\delta M + (2A/a)\sin(M - M_0) \tag{4}$$

where δ, M_0, A, and λ_0 are all integration constants, and $M = nt$, the mean anomaly. The first-order correction to the orbital energy is

$$\mathcal{E} = (\mu/a)\left[-\tfrac{1}{2} + 2s/a + \dot{\epsilon}/n\right] = -(\mu/2a)(1 - 2\delta) \tag{5}$$

In these orbital expansions we have freedom to choose the radius a about which we expand. If we fix this radius as the radius of a circular orbit of the same orbital energy, then $\delta = 0$. These equations then describe epicyclic motion. We can summarize the solution as

$$r = a - A\cos(M - M_0)$$

$$\nu = (M - M_0) + (2A/a)\sin(M - M_0), \qquad \lambda = \nu + \omega \tag{6}$$

where ν is the true anomaly and $\nu = 0$ when r reaches its minimum value. The constant ω is the argument of perigee, and A is the magnitude of the radial variation and is called the epicycle amplitude. We note that in the preceding equations the constants a and n still satisfy $a^3n^2 = \mu$.

It is not difficult to expand this epicycle equations to three-dimensional Keplerian motion. This introduces the orbital elements of inclination I and longitude of the ascending node Ω, which are both constants. When we consider three-dimensional motion, then the direction of $\lambda = 0$ can be defined. When the constant ω is associated with the argument of perigee, then λ can be identified as the argument of latitude, the angle measured from the ascending node to the satellite on the orbital plane.

In the case when $I = 0$, then $\Omega = 0$ and the angle λ is still defined on the orbital plane and is measured from the position of the satellite at some fixed time t_0 that satisfies $M_0 = nt_0$. If the epicycle amplitude vanishes, then $\omega = 0$ and λ reduces to the true anomaly. It is not convenient to use the true anomaly as perigee becomes undefined for very small eccentricities. For an inclined circular orbit, it is more useful to measure the phase from the ascending node, which is also straightforward to determine onboard a satellite. This complicates our description because from the first of Eqs. (6) we have associated M_0 with perigee passage. Letting the mean anomaly at the ascending node be M_e, then to first order in the epicycle amplitude, we can express the orbit as

$$r = a - A\cos(\alpha - \alpha_P)$$

$$\lambda = \alpha + (2A/a)[\sin(\alpha - \alpha_P) + \sin\alpha_P]$$

$$I = I_0, \qquad \Omega = \Omega_0, \qquad \alpha = M - M_e \tag{7}$$

The angle α is proportional to time and varies through 2π in one orbit. We refer to this angle as the epicycle phase. When $\alpha = 0$, then $\lambda = 0$, which is at the initial ascending node, and when $\alpha = \alpha_P \equiv M_0 - M_e$, then the satellite is at the perigee passage. The argument of perigee is related to M_e or α_P through

$$\omega = \alpha_P + (2A/a)\sin\alpha_P \tag{8}$$

For non inclined orbits, we can set $M_e = M_0$, and hence, α is measured from perigee passage.

The representation of the location of the satellite that we have used is redundant in that α is a measure of time, but there are four coordinates for the satellite instead of three. We shall keep this system because the evolution of the orbital plane and the satellite's location on that plane are readily expressed in these variables. We note that this representation is a mixture of two osculating orbital elements and two coordinates.

Perturbed Motion

Once perturbing terms are included in the geopotential, these basic equations are generalized to the following form, which describes the perturbed motion of a satellite about an axisymmetric planet. In the rest of this paper we derive expressions for the various terms appearing in these equations.

To describe the perturbed motion of a satellite, it becomes important to clarify the initial conditions or coordinates of equations. We have defined the epicycle phase α by

$$\alpha = nt \tag{9}$$

where $t = 0$ when the satellite initially crosses an ascending node. We can define the values of I_0 and Ω_0 to describe the instantaneous orbital plane at $\lambda = \alpha = 0$ (or $t = 0$), which are osculating inclination and ascending node. We have chosen these values throughout our formulations. This definition imposes the condition that all of the small correction terms on inclination, ascending node, and argument of latitude due to the perturbation should vanish at $\alpha = 0$. We have defined our mean semimajor axis a to satisfy

$$\mathcal{E} \equiv -(\mu/2a) \tag{10}$$

where $\mathcal{E}$ is the total orbital energy of a satellite that is constant under an axisymmetric potential. Note that this definition of mean semimajor axis is also applied by Douglas and Ingram.[5] Then the epicycle frequency n is defined through $a^3n^2 = \mu$. Under these definitions, we will show that the perturbed motion of a satellite about an axisymmetric planet can be described by

$$r = a(1 + \varrho) - A\cos(\alpha - \alpha_P) + a\chi\sin[(1 + \kappa)\alpha] + \Delta_r$$

$$I = I_0 + \Delta_I, \qquad \Omega = \Omega_0 + \vartheta\alpha + \Delta_\Omega$$

$$\lambda = (1 + \kappa)\alpha + (2A/a)[\sin(\alpha - \alpha_P) + \sin\alpha_P]$$

$$- 2\chi\{1 - \cos[(1 + \kappa)\alpha]\} + \Delta_\lambda \tag{11}$$

The secular variations are described by the quantities ϱ, ϑ, and κ. The first of these describes a constant offset in the mean orbital radius due to the extra terms in the potential. This term can be removed completely from the equations by a suitable redefinition of the quantity a. The secular change in the ascending node is described by ϑ, which gives a linear variation of Ω with time. The secular drift in the argument of latitude is described by κ. The long-periodic variations in the orbit are described by χ and the short-period variations are expressed as a Fourier series through the terms Δ_x for each of the four coordinates. When the epicycle phase changes by 2π with the frequency n (the anomalistic frequency), which is derived by our definition of mean semimajor axis, then the argument of latitude changes by an amount of $2\pi + 2\kappa\pi$. The argument of latitude can then be considered to vary with time as $\lambda \sim n_N t$ where

$$n_N = (1 + \kappa)n \tag{12}$$

is the nodal frequency.

Let a_0 be the osculating semimajor axis at an arbitrary time t, then the first-order approximation on our mean semimajor axis a in terms of a_0 is given by

$$-\frac{\mu}{2a} = -\frac{\mu}{2a_0} + \frac{\mu}{a_0}\sum_m J_m\left(\frac{R}{a}\right)^m P_m(\sin I_0 \sin\alpha) \tag{13}$$

from which we derive

$$a_0 = a\left[1 - 2\sum_m J_m\left(\frac{R}{a}\right)^m P_m(\sin I_0 \sin\alpha)\right] \tag{14}$$

Hence, the osculating mean motion n_0 in terms of our mean motion n is also expressed by

$$n_0 = n\left[1 + 3\sum_m J_m\left(\frac{R}{a}\right)^m P_m(\sin I_0 \sin\alpha)\right] \tag{15}$$

through $a^3 n^2 = a_0^3 n_0^2 = \mu$. Taking the orbital average of these osculating elements, we have

$$\bar{a}_0 = a(1 - 2\rho), \qquad \bar{n}_0 = n(1 + 3\rho) \tag{16}$$

with

$$\rho = \sum_m \frac{\varrho_{2m}}{2m - 1} \tag{17}$$

where ϱ_{2m} is the contribution to ϱ in Eq. (11) by even zonal perturbations. Explicit expression of ϱ_{2m} will be shown in Sec. IV, and then we will find that the mean motion defined by Kozai[1] has the same expression as in Eq. (16) when J_2 perturbation is considered.

The two anomalistic and nodal frequencies discussed earlier are rewritten in terms of $\bar{n}_0$ by

$$n_A = \bar{n}_0(1 - 3\rho), \qquad n_N = \bar{n}_0(1 - 3\rho + \kappa) \tag{18}$$

If we describe the anomalistic frequency in terms of osculating mean motion n_0 at the initial perigee (hence, $\alpha = \alpha_P$) and the nodal frequency at the initial node ($\alpha = 0$) for the J_2 case, both frequencies give the results derived by Moe and Karp[9] (for anomalistic frequency) and by Merson[10] and Blitzer[11] (for nodal frequency), if e^2 and $J_2 e$ terms are neglected.

In the rest of this paper, we shall derive expressions for the perturbative terms appearing in Eq. (11).

III. Linearized Equations of Motion

In this section we derive the linearized equations of motion describing the motion of a satellite in three dimensions using the redundant representation developed in the preceding section. We restrict ourselves to an axisymmetric model of the Earth because these terms provide the principal perturbation sources.

Equations of Motion in New Variables

When started from spherical polar coordinates (r, θ, φ), the equations of motion around an oblate Earth are

$$\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) = -\frac{\partial V}{\partial r}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\varphi}^2 = -\frac{\partial V}{\partial \theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\sin^2\theta\dot{\varphi}) = 0 \tag{19}$$

where the potential V is defined by

$$V = -\frac{\mu}{r}\left[1 - \sum_m J_m\left(\frac{R}{r}\right)^m P_m(\cos\theta)\right] \tag{20}$$

and R is the Earth's mean equatorial radius. Note that in this formulation the coefficient J_2 is positive.

We immediately have an integral of the third equation:

$$H_z = r^2\sin^2\theta\dot{\varphi} \tag{21}$$

which is the z component of angular momentum and is a constant of the motion. We can relate the position of the satellite (θ, φ) to the orbital parameters (I, Ω, λ) such that

$$\sin\theta\cos(\varphi - \Omega) = \cos\lambda$$

$$\sin\theta\sin(\varphi - \Omega) = \cos I\sin\lambda, \qquad \cos\theta = \sin I\sin\lambda \tag{22}$$

We define our instantaneous orbital plane (I, Ω) to contain the velocity vector and the position vector. Here $\boldsymbol{e}_J$ is the vector normal to this plane, which is given by

$$\boldsymbol{e}_J = \sin I\sin\Omega\boldsymbol{i} - \sin I\cos\Omega\boldsymbol{j} + \cos I\boldsymbol{k} \tag{23}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are basis vectors of an inertial Cartesian frame. We may write the velocity vector $\boldsymbol{v}$ in terms of spherical polar components:

$$\boldsymbol{v} = \frac{\mathrm{d}(r\sin\theta\cos\varphi)}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}(r\sin\theta\sin\varphi)}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}(r\cos\theta)}{\mathrm{d}t}\boldsymbol{k} \tag{24}$$

By the solving of $\boldsymbol{v}\cdot\boldsymbol{e}_J = 0$, the condition for the velocity vector to be confined to the (I, Ω) plane is

$$\dot{\theta}\cos I + \dot{\varphi}\sin\theta\sin I\cos\lambda = 0 \tag{25}$$

If the true anomaly is ν, the instantaneous angular momentum of the satellite is $r^2\dot{\nu}$, and its component in the z direction is

$$H_z = r^2\dot{\nu}\cos I \tag{26}$$

Hence, equating this to Eq. (21), we get

$$\sin^2\theta\dot{\varphi} = \cos I\dot{\nu} \tag{27}$$

Condition (25) then requires

$$\sin\theta\dot{\theta} = -\sin I\cos\lambda\dot{\nu} \tag{28}$$

If we square and add both Eqs. (27) and (28), we have

$$\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 = \dot{\nu}^2 \tag{29}$$

Other useful relations can be found by combining the time derivatives of the first and the last of Eqs. (22) and eliminating $\dot{\lambda}$:

$$\dot{I}\sin\lambda = \dot{\Omega}\sin I\cos\lambda \tag{30}$$

where we have used Eqs. (27) and (28). Note that this equality can also be found from the Gauss planetary equations.[12] Then by substituting from $\dot{\theta}$ and $\dot{I}$ to $\dot{\nu}$ and $\dot{\Omega}$, the derivative of the last of Eqs. (22) gives

$$\dot{\nu} = \dot{\lambda} + \dot{\Omega}\cos I \tag{31}$$

The middle of Eqs. (22) does not contain any further information.

If we multiply through the second of Eqs. (19) by $\sin\theta$, we can write it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\nu}\sin I\cos\lambda) + r^2\dot{\nu}^2\sin I\sin\lambda = -(1 - \sin^2 I\sin^2\lambda)\frac{\partial V}{\partial(\sin I\sin\lambda)} \tag{32}$$

Using Eqs. (26) and (30), we can write this equation as

$$H_z\dot{I} = -\cos^2 I\cos\lambda\frac{\partial V}{\partial(\sin I\sin\lambda)} \tag{33}$$

For an axisymmetric potential, the total orbital energy $\mathcal{E}$ is conserved

$$\mathcal{E} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\nu}^2) + V \tag{34}$$

When $\dot{v}^2$ is eliminated using Eqs. (29) and (34), finally the first of Eqs. (19) can be written as

$$\ddot{r} - \frac{2\mathcal{E} - \dot{r}^2}{r} = -\frac{2V}{r} - \frac{\partial V}{\partial r} \tag{35}$$

We can summarize all of the equations of motion as

$$\ddot{r} - \frac{2\mathcal{E} - \dot{r}^2}{r} - \frac{\mu}{r^2} = \frac{\mu}{r^2}\sum_m (m-1)J_m\left(\frac{R}{r}\right)^m P_m(\sin I \sin\lambda)$$

$$H_z\dot{I} = -\frac{\mu}{r}\sum_m J_m\left(\frac{R}{r}\right)^m \cos^2 I \cos\lambda \frac{\mathrm{d}P_m(\sin I \sin\lambda)}{\mathrm{d}(\sin I \sin\lambda)}$$

$$\dot{\Omega}\sin I \cos\lambda = \dot{I}\sin\lambda, \qquad H_z = r^2\dot{v}\cos I$$

$$\dot{\lambda} = \dot{v} - \dot{\Omega}\cos I \tag{36}$$

where H_z and $\mathcal{E}$ are constants of the motion.

First-Order Linearized Equations

We seek solutions to the equations in the neighbourhood of circular orbits. Note that inclined circular orbits do not exist as solutions of the perturbed equations. The deviations from circular orbits are considered to be $\mathcal{O}(J_2)$. We linearize Eqs. (36) based on the epicycle description of the motion as deviations about the orbit: $r = a$ and $\lambda = \alpha$. Because the perturbing terms on the right-hand side of Eqs. (36) are in the form of a sum over harmonics, the deviations of the satellite coordinates can also be expressed as a linear sum of terms. We can then consider each term individually. The response to the mth harmonic will then be

$$r = a + s_m, \qquad I = I_0 + \iota_m$$

$$\Omega = \Omega_0 + o_m, \qquad \lambda = \alpha + \epsilon_m \tag{37}$$

The variation in H_z can be expressed as $H_z = a^2 n(1 + \delta_m)\cos I_0$, where

$$\delta_m = 2s_m/a - \iota_m \tan I_0 + \dot{\epsilon}_m/n + (\dot{o}_m/n)\cos I_0 \tag{38}$$

and must be constant. Linearizing the third of Eqs. (36) gives

$$\dot{o}_m \sin I_0 = \dot{\iota}_m \tan\alpha \tag{39}$$

When $\mathcal{O}(J_m^2)$ terms are ignored, the second of Eqs. (36) becomes

$$\frac{\dot{\iota}_m}{n} = -J_m\left(\frac{R}{a}\right)^m \cos I_0 \cos\alpha \frac{\mathrm{d}P_m(\sin I_0 \sin\alpha)}{\mathrm{d}(\sin I_0 \sin\alpha)} \tag{40}$$

and the linearized form of the first of Eqs. (36) is, applying our definition of semimajor axis (10),

$$\ddot{s}_m \big/ an^2 + s_m/a = (m-1)J_m(R/a)^m P_m(\sin I_0 \sin\alpha) \tag{41}$$

This completes the linearization of the equations we need to solve.

Before proceeding to solve these equations, consider the first-order approximation to the total orbital energy, which should be constant. From Eq. (34),

$$\mathcal{E} = -\mu/2a + (\mu/a)\big[2s_m/a + \dot{\epsilon}_m/n$$
$$+ (\dot{o}_m/n)\cos I_0 + J_m(R/a)^m P_m(\sin I_0 \sin\alpha)\big] \tag{42}$$

However, using Eq. (38) we can express this in terms of δ_m

$$\mathcal{E} = -\mu/2a + (\mu/a)\big[\delta_m + \iota_m \tan I_0 + J_m(R/a)^m P_m(\sin I_0 \sin\alpha)\big] \tag{43}$$

From our definition of the semimajor axis, the term in the square brackets must vanish.

IV. Solving the Linearized Equations

In this section, we shall solve Eqs. (38–41) ignoring $\mathcal{O}(J_m^2)$ terms as well as $\mathcal{O}(J_m e)$ and $\mathcal{O}(J_m J_l)$ coupling terms. We shall treat the even and odd zonal perturbations separately. It will be convenient to introduce the following notations:

$$x' = \frac{\mathrm{d}x}{\mathrm{d}\alpha} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{\dot{x}}{n}, \qquad A_m = J_m\left(\frac{R}{a}\right)^m, \qquad e = \frac{A}{a} \tag{44}$$

where e is the normalized, dimensionless, epicycle amplitude or conventionally eccentricity.

Solution for the Even Zonal Harmonics

If m is even, then replacing m by $2m$ and integrating Eq. (40) gives

$$\iota_{2m} = -A_{2m}\cot I_0[P_{2m}(\sin I_0 \sin\alpha) - P_{2m}(0)] \tag{45}$$

The integration constant is determined to satisfy $\iota_{2m} = 0$ when $\alpha = 0$ following our definition of I_0. When the solution for ι_{2m} is used in Eq. (43), $\mathcal{E}$ is constant as expected. Also δ_{2m} is constant and can be chosen to be

$$\delta_{2m} = -A_{2m}P_{2m}(0) \tag{46}$$

to satisfy our definition of mean semimajor axis. When the addition theorem for the Legendre function is used, the ι_{2m} solution may also be expressed as

$$\iota_{2m} = 2A_{2m}\cot I_0 \sum_{l=1}^{m}(-1)^l L_{2m}^{2l}(1 - \cos 2l\alpha) \tag{47}$$

where the coefficients L_i^j are given by

$$L_i^j = \frac{(i-j)!}{(i+j)!}P_i^j(0)P_i^j(\cos I_0) \tag{48}$$

When Eq. (47) is used in Eq. (39), the equation for the ascending node perturbation becomes

$$o'_{2m} = 2A_{2m}\frac{\cot I_0}{\sin I_0}\sum_{l=1}^{m}(-1)^l 2l L_{2m}^{2l}\frac{\sin\alpha \sin 2l\alpha}{\cos\alpha} \tag{49}$$

This equation can be integrated using the relation

$$(-1)^l \frac{\cos(2l+1)\alpha}{\cos\alpha} = 1 + 2\sum_{k=1}^{l}(-1)^k \cos 2k\alpha \tag{50}$$

to give the o_{2m} solution:

$$o_{2m} = -2A_{2m}\frac{\cot I_0}{\sin I_0}\sum_{l=1}^{m}\left[2lL_{2m}^{2l}\alpha + (-1)^l \Lambda_{2m}^{2l}\sin 2l\alpha\right] \tag{51}$$

where the notation of Λ_{2m}^{2l} is defined by

$$\Lambda_{2m}^{2l} = L_{2m}^{2l} + 2\sum_{k=l+1}^{m}\frac{2k}{2l}L_{2m}^{2k} \tag{52}$$

Equation (41) becomes

$$\frac{s''_{2m}}{a} + \frac{s_{2m}}{a} = (2m-1)A_{2m}\left[L_{2m}^0 + 2\sum_{l=1}^{m}(-1)^l L_{2m}^{2l}\cos 2l\alpha\right] \tag{53}$$

and integrates to yield

$$\frac{s_{2m}}{a} = -e\cos(\alpha - \alpha_P) + (2m-1)A_{2m}$$
$$\times\left[L_{2m}^0 + 2\sum_{l=1}^{m}\frac{(-1)^l}{1-(2l)^2}L_{2m}^{2l}\cos 2l\alpha\right] \tag{54}$$

where e and α_P are integration constants. By substituting these solutions into Eq. (38) and integrating, we find

$$\epsilon_{2m} = -o_{2m}\cos I_0 + 2e[\sin(\alpha - \alpha_P) + \sin\alpha_P] - A_{2m}\left\{(4m-1)L_{2m}^0\alpha + 2\sum_{l=1}^{m}\frac{(-1)^l}{2l}\times\left[1 + \frac{2(2m-1)}{1-(2l)^2}\right]L_{2m}^{2l}\sin 2l\alpha\right\} \tag{55}$$

There is a secular drift in this result, which we have already identified with the nodal period n_N. The contribution of the J_{2m} term to κ can be identified as the coefficient of α in Eq. (55). We can similarly define the contribution to the other secular terms in our solution. If we denote these contributions by a subscript of $2m$, then

$$\varrho_{2m} = (2m-1)A_{2m}L_{2m}^0$$

$$\vartheta_{2m} = -2A_{2m}\frac{\cot I_0}{\sin I_0}\sum_{l=1}^{m}2lL_{2m}^{2l}$$

$$\kappa_{2m} = -\vartheta_{2m}\cos I_0 - (4m-1)A_{2m}L_{2m}^0 \tag{56}$$

There are no long-period variations arising from the even zonal harmonics; hence,

$$\chi_{2m} = 0 \tag{57}$$

and the short-periodic perturbations due to J_{2m} are

$$\Delta_{r_{2m}} = \sum_{l=1}^{m}\Delta r_{2m}^l\cos 2l\alpha, \qquad \Delta_{I_{2m}} = \sum_{l=1}^{m}\Delta I_{2m}^l(1-\cos 2l\alpha)$$

$$\Delta_{\Omega_{2m}} = \sum_{l=1}^{m}\Delta\Omega_{2m}^l\sin 2l\alpha, \qquad \Delta_{\lambda_{2m}} = \sum_{l=1}^{m}\Delta\lambda_{2m}^l\sin 2l\alpha \tag{58}$$

where

$$\Delta r_{2m}^l = 2(2m-1)A_{2m}\frac{(-1)^l}{1-(2l)^2}L_{2m}^{2l}a$$

$$\Delta I_{2m}^l = 2A_{2m}(-1)^l\cot I_0 L_{2m}^{2l}$$

$$\Delta\Omega_{2m}^l = -2A_{2m}(-1)^l\frac{\cot I_0}{\sin I_0}\Lambda_{2m}^{2l}$$

$$\Delta\lambda_{2m}^l = -\Delta\Omega_{2m}^l\cos I_0 - 2A_{2m}\frac{(-1)^l}{2l}\left[1 + \frac{2(2m-1)}{1-(2l)^2}\right]L_{2m}^{2l} \tag{59}$$

Solutions for the Odd Zonal Harmonics

We next consider the perturbation correction terms due to the odd zonal harmonics and replace m by $2m+1$. The procedure for determining the perturbations to the inclination and ascending node is the same as in the preceding section. They are found to be

$$\iota_{2m+1} = -2A_{2m+1}\cot I_0\sum_{l=0}^{m}(-1)^l L_{2m+1}^{2l+1}\sin(2l+1)\alpha$$

$$o_{2m+1} = -2A_{2m+1}\frac{\cot I_0}{\sin I_0}\sum_{l=0}^{m}(-1)^l\Lambda_{2m+1}^{2l+1}[1-\cos(2l+1)\alpha] \tag{60}$$

where

$$\Lambda_{2m+1}^{2l+1} = L_{2m+1}^{2l+1} + 2\sum_{k=l+1}^{m}\frac{2k+1}{2l+1}L_{2m+1}^{2k+1} \tag{61}$$

Our choice for the constant δ_{2m+1} follows from using Eq. (60) in Eq. (43), which yields

$$\delta_{2m+1} = -A_{2m+1}P_{2m+1}(0) = 0 \tag{62}$$

We next consider the radial equation for s_{2m+1},

$$\frac{s_{2m+1}''}{a} + \frac{s_{2m+1}}{a} = 4mA_{2m+1}\sum_{l=0}^{m}L_{2m+1}^{2l+1}(-1)^l\sin(2l+1)\alpha \tag{63}$$

Unlike the even zonal case, the right-hand side of this equation contains terms in $\sin\alpha$. This leads to solutions of the form $\alpha\cos\alpha$, which does not properly reflect the long-period nature of the solution. It adequately describes only the initial part of the long-period variation. We can see that the solutions are long periodic by noting that this equation is linked with the perturbation to the argument of latitude. Both these terms should then vary with the nodal frequency n_N rather than n itself. If we let $\beta = (1+\kappa)\alpha$, then the coupled equations for the perturbed position of the satellite on the plane become

$$s_{2m+1}''/a + s_{2m+1}/a = K_0\sin\beta$$

$$2s_{2m+1}/a + \epsilon_{2m+1}' = K_1\sin\beta \tag{64}$$

where

$$K_0 = 4mA_{2m+1}L_{2m+1}^1$$

$$K_1 = -2A_{2m+1}\left(L_{2m+1}^1 - \Lambda_{2m+1}^1\cot^2 I_0\right) \tag{65}$$

We can integrate directly the first of these equations, and ignoring the complementary function (which is just the epicycle term already accounted for) and ensuring the periodic perturbation in ϵ_{2m+1} vanishes at $\alpha = 0$ gives:

$$s_{2m+1} = -(K_0/2\kappa)a\sin\beta$$

$$\epsilon_{2m+1} = (K_0/\kappa + K_1)(1-\cos\beta) \tag{66}$$

These equations contain $\mathcal{O}(J_{2m})$ terms in the denominator for which reason we are keeping β in these terms. Note that for other terms whose coefficients are $\mathcal{O}(J_{2m+1})$ (without the κ denominator) we can use both α and β angles. In principal we can keep all of these terms, but in practice we only ever need to consider the J_2 term:

$$\kappa_2 = A_2\left(-3L_2^0 + 4\cot^2 I_0 L_2^2\right) = \tfrac{3}{4}A_2\left(4 - 5\sin^2 I_0\right) \tag{67}$$

This result is adequate except at the critical inclination where $\sin^2 I = \frac{4}{5}$. For satellite orbits with inclinations in this vicinity, κ_4 and higher terms may need to be considered. The rest of the terms in the s_{2m+1} and ϵ_{2m+1} equations are similarly integrated as in the preceding section. Omitting the epicycle terms, we formulate the s_{2m+1} and ϵ_{2m+1} solutions as follows:

$$s_{2m+1} = a\chi_{2m+1}\sin\beta + \Delta_{r_{2m+1}}$$

$$\epsilon_{2m+1} = -2\chi_{2m+1}(1-\cos\beta) + \Delta_{\lambda_{2m+1}} \tag{68}$$

where the long-period terms are given by

$$\chi_{2m+1} = -(2mA_{2m+1}/\kappa_2)L_{2m+1}^1 \tag{69}$$

and we have replaced κ by κ_2. Notice that our χ term [Eq. (69)] gives exactly the same result as Cook introduced.[4]

The short-periodic terms then take the form

$$\Delta_{r_{2m+1}} = \sum_{l=1}^{m}\Delta r_{2m+1}^l\sin(2l+1)\alpha$$

$$\Delta_{I_{2m+1}} = \sum_{l=0}^{m}\Delta I_{2m+1}^l\sin(2l+1)\alpha$$

$$\Delta_{\Omega_{2m+1}} = \sum_{l=0}^{m}\Delta\Omega_{2m+1}^l[1-\cos(2l+1)\alpha]$$

$$\Delta_{\lambda_{2m+1}} = \sum_{l=0}^{m}\Delta\lambda_{2m+1}^l[1-\cos(2l+1)\alpha] \tag{70}$$

where:

$$\Delta r^{l}_{2m+1} = 4mA_{2m+1}\frac{(-1)^l}{1-(2l+1)^2}L^{2l+1}_{2m+1}a$$

$$\Delta I^{l}_{2m+1} = -2A_{2m+1}(-1)^l \cot I_0 L^{2l+1}_{2m+1}$$

$$\Delta \Omega^{l}_{2m+1} = -2A_{2m+1}(-1)^l \frac{\cot I_0}{\sin I_0}\Lambda^{2l+1}_{2m+1}$$

$$\Delta \lambda^{0}_{2m+1} = -\Delta\Omega^{0}_{2m+1}\cos I_0 - 2A_{2m+1}L^{1}_{2m+1}$$

$$\Delta \lambda^{l}_{2m+1} = -\Delta\Omega^{l}_{2m+1}\cos I_0 - 2A_{2m+1}\frac{(-1)^l}{2l+1}\left[1+\frac{4m}{1-(2l+1)^2}\right]L^{2l+1}_{2m+1} \tag{71}$$

We note that in Eq. (70) each sum starts from $l = 0$ except for the radial equation, where it starts from $l = 1$. There are no secular effects arising from the odd zonal harmonics; hence,

$$\varrho_{2m+1} = \vartheta_{2m+1} = \kappa_{2m+1} = 0 \tag{72}$$

V. Postepicycle Description of Keplerian Motion

Recalling the discussion in Sec. II, now we consider the second-order effects of a slightly eccentric orbit. Assume $r = a + s + s_s$ and $\lambda = \alpha + \epsilon + \epsilon_s$, where notations of x_s are the second-order (postepicyclic) corrections and the first-order corrections of s and ϵ have been given by Eq. (7). Using the relation $\delta = 2s/a + \epsilon' = 0$, which comes from our definition of the mean semimajor axis, we can expand Eqs. (1) to second order:

$$\frac{s''_s}{a} + \frac{s_s}{a} - 2\left(\frac{2s_s}{a} + \epsilon'_s\right) = -5\left(\frac{s}{a}\right)^2 + \frac{1}{2}(\epsilon')^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2s_s}{a} + \epsilon'_s - \left(\frac{s}{a}\right)^2 - \frac{1}{2}(\epsilon')^2\right] = 0 \tag{73}$$

The second of Eqs. (73) can directly be integrated to give

$$\delta_s = 2s_s/a + \epsilon'_s - (s/a)^2 - \tfrac{1}{2}(\epsilon')^2 \tag{74}$$

where δ_s is the second-order constant of integration. Using this δ_s constant, the orbital energy $\mathcal{E}$ can be written as

$$\mathcal{E} = -(\mu/2a)\left[1 - 2\delta_s + 3(s/a)^2 - (s'/a)^2 - (\epsilon')^2\right]$$

$$= -(\mu/2a)\left[1 - 2\delta_s - e^2\right] \tag{75}$$

Therefore, we can choose δ_s to be $-e^2/2$ to leave the orbital energy unchanged. This effectively determines our choice of a. We can now derive the postepicycle terms by solving Eqs. (73) to obtain

$$s_s/a = \tfrac{1}{2}e^2[1 - \cos(2\alpha - 2\alpha_P)]$$

$$\epsilon_s = \tfrac{5}{4}e^2[\sin(2\alpha - 2\alpha_P) + \sin 2\alpha_P] \tag{76}$$

We can formulate the postepicyclic expression of a near circular orbit such that

$$r = a - ae\cos(\alpha - \alpha_P) + \tfrac{1}{2}ae^2[1 - \cos(2\alpha - 2\alpha_P)]$$

$$\lambda = \alpha + 2e[\sin(\alpha - \alpha_P) + \sin\alpha_P] + \tfrac{5}{4}e^2[\sin(2\alpha - 2\alpha_P) + \sin 2\alpha_P] \tag{77}$$

VI. J_2 Second-Order Expansion

Using the J_2 first-order solutions, we can expand the J_2 perturbation solution up to second order. We have to approximate the equations of motion, including the J_2 potential, by taking account of second-order effects. Assume

$$r = a + s_2 + s_{2s}, \qquad I = I_0 + \iota_2 + \iota_{2s}$$

$$\Omega = \Omega_0 + o_2 + o_{2s}, \qquad \lambda = \alpha + \epsilon_2 + \epsilon_{2s} \tag{78}$$

where the notations of x_{2s} represent $\mathcal{O}(J_2^2)$ corrections. The first-order corrections x_2 are obtained by

$$s_2 = -ae\cos(\alpha - \alpha_P) + a\varrho_2 + \Delta r^1_2 \cos 2\alpha$$

$$\iota_2 = \Delta I^1_2(1 - \cos 2\alpha), \qquad o_2 = \vartheta_2\alpha + \Delta\Omega^1_2 \sin 2\alpha$$

$$\epsilon_2 = 2e[\sin(\alpha - \alpha_P) + \sin\alpha_P] + \kappa_2\alpha + \Delta\lambda^1_2 \sin 2\alpha \tag{79}$$

where all coefficients are introduced in Appendix A.

J_2 Second-Order Equation

We now expand the fourth of Eqs. (36) up to second order. Similarly we can define a J_2^2-order constant δ_{2s} such that $H_z = a^2n(1 + \delta_2 + \delta_{2s})\cos I_0$, and then δ_{2s} is given by

$$\delta_{2s} = 2s_{2s}/a - \iota_{2s}\tan I_0 + \epsilon'_{2s} + o'_{2s}\cos I_0 + (2s_2/a - \iota_2\tan I_0)(\delta_2 + \iota_2\tan I_0) - o'_2\iota_2\sin I_0 - \iota_2^2\big/2 - 3(s_2/a)^2 \tag{80}$$

We also note the following equality for the first-order δ_2 constant:

$$\delta_2 + \iota_2\tan I_0 + A_2P_2(\sin I_0\sin\alpha) = 0 \tag{81}$$

and which may frequently be used to arrange some results introduced in this section. We introduce notations of $z = \sin I_0\sin\alpha$ and $\Delta z = \iota_2\cos I_0\sin\alpha + \epsilon_2\sin I_0\cos\alpha$, then $\sin I\sin\lambda$ is approximated by $z + \Delta z$, and the Legendre functions can be expanded by

$$P_2(\sin I\sin\lambda) = P_2(z) + \frac{\mathrm{d}P_2(z)}{\mathrm{d}z}\Delta z + \mathcal{O}(\Delta z^2)$$

$$\frac{\mathrm{d}P_2(\sin I\sin\lambda)}{\mathrm{d}(\sin I\sin\lambda)} = \frac{\mathrm{d}P_2(z)}{\mathrm{d}z} + \frac{\mathrm{d}^2P_2(z)}{\mathrm{d}z^2}\Delta z + \mathcal{O}(\Delta z^2) \tag{82}$$

The second and third of Eqs. (36) are expanded to second order by

$$\iota'_{2s} = A_2\cos I_0\cos\alpha\left[(y_2 + \epsilon_2\tan\alpha)\frac{\mathrm{d}P_2(z)}{\mathrm{d}z} - \frac{\mathrm{d}^2P_2(z)}{\mathrm{d}z^2}\Delta z\right]$$

$$o'_{2s} = A_2\cot I_0\sin\alpha\left[(y_2 + \iota_2\cot I_0 - \epsilon_2\cot\alpha)\frac{\mathrm{d}P_2(z)}{\mathrm{d}z} - \frac{\mathrm{d}^2P_2(z)}{\mathrm{d}z^2}\Delta z\right]$$

$$y_2 = \delta_2 + \frac{3s_2}{a} + 2\iota_2\tan I_0 \tag{83}$$

Finally, the first of Eqs. (36) becomes

$$\frac{s''_{2s}}{a} + \frac{s_{2s}}{a} = 2\left(\frac{s'_2}{a}\right)^2 - \left(\frac{s_2}{a}\right)^2 - A_2\left[\frac{4s_2}{a}P_2(z) - \frac{\mathrm{d}P_2(z)}{\mathrm{d}z}\Delta z\right] \tag{84}$$

The orbital energy $\mathcal{E}$ is also expanded to second order and yields

$$\mathcal{E} = -(\mu/2a)(1 - \varepsilon) \tag{85}$$

where

$$\varepsilon = 2\delta_{2s} + 2\iota_{2s}\tan I_0 + \left[\left(\frac{s'_2}{a}\right)^2 + \left(\frac{s_2}{a}\right)^2\right] + (\delta_2 + 2\iota_2\tan I_0)^2 + \left(1 - \tan^2 I_0\right)\iota_2^2 - 2A_2\left[\frac{s_2}{a}P_2(z) - \frac{\mathrm{d}P_2(z)}{\mathrm{d}z}\Delta z\right] \tag{86}$$

From our definition of the semimajor axis, ε must vanish. Therefore, by eliminating $\delta_{2s} + \iota_{2s}\tan I_0$ from $\varepsilon = 0$ and Eq. (80), we obtain the ϵ_{2s} equation as follows:

$$(\epsilon_{2s} + o_{2s}\cos I_0)' = -\frac{2s_{2s}}{a} + \frac{5}{2}\left(\frac{s'_2}{a}\right)^2 - \frac{1}{2}\left(\frac{s_2}{a}\right)^2 + o'_2\iota_2\sin I_0 + A_2\left[\frac{3s_2}{a}P_2(z) - \frac{1}{2}A_2\{P_2(z)\}^2 - \frac{\mathrm{d}P_2(z)}{\mathrm{d}z}\Delta z\right] \tag{87}$$

Once o_{2s} and s_{2s} have been found by solving Eqs. (83) and (84), the final unknown ϵ_{2s} can be solved for using Eq. (87).

J_2 Second-Order Solution

Substituting the J_2 first-order solutions into Eqs. (83), we can first integrate to obtain the ι_{2s} and o_{2s} solutions. We use the following expressions to derive these solutions:

$$\frac{\mathrm{d}P_2(z)}{\mathrm{d}z} = \frac{8L_2^2}{\sin I_0}\sin\alpha, \qquad \frac{\mathrm{d}^2P_2(z)}{\mathrm{d}z^2} = \frac{8L_2^2}{\sin^2 I_0} \tag{88}$$

The ι_{2s} correction is found by direct integration,

$$\iota_{2s} = \iota_{2s}(\alpha) - \iota_{2s}(0)$$

$$\iota_{2s}(\alpha) = \Delta I_{2s}^1\cos 2\alpha + \Delta I_{2s}^2\cos 4\alpha - 4A_2L_2^2\cot I_0\Big\{\kappa_2\alpha\sin 2\alpha + e\Big[2\sin\alpha_P\sin 2\alpha + \tfrac{1}{2}\cos(\alpha+\alpha_P) - \tfrac{7}{6}\cos(3\alpha-\alpha_P)\Big]\Big\} \tag{89}$$

where the J_2^2 coefficients are

$$\Delta I_{2s}^1 = 2A_2^2L_2^2\cot I_0\Big[L_2^0 + 2\big(1-3\cot^2 I_0\big)L_2^2\Big]$$

$$\Delta I_{2s}^2 = -\tfrac{1}{3}A_2^2\big(L_2^2\big)^2\cot I_0\big(8+3\cot^2 I_0\big) \tag{90}$$

By combining $\kappa_2\alpha\sin 2\alpha$ and $\mathcal{O}(J_2e)$ terms in Eq. (89) with the J_2 first-order inclination solution (47), we can rearrange the ι_2 and ι_{2s} solutions such that

$$\iota_2 = \Delta I_2^1(1-\cos 2\gamma)$$

$$\iota_{2s}(\alpha) = \iota_{2e}(\alpha) + \Delta I_{2s}^1\cos 2\alpha + \Delta I_{2s}^2\cos 4\alpha$$

$$\iota_{2e}(\alpha) = \tfrac{3}{8}A_2e\sin 2I_0\Big[\cos(\alpha+\alpha_P) + \tfrac{1}{3}\cos(3\alpha-\alpha_P)\Big] \tag{91}$$

where the notation x_{2e} represents the $\mathcal{O}(J_2e)$ solution and the γ angle is defined by

$$\gamma = (1+\kappa_2)\alpha + 2e[\sin(\alpha-\alpha_P) + \sin\alpha_P] \tag{92}$$

Similarly, we can derive the o_{2s} solution, which is

$$o_{2s} = o_{2s}(\alpha) - o_{2s}(0)$$

$$o_{2s}(\alpha) = (\vartheta_2\kappa_2 + \vartheta_{2s})\alpha + \Delta\Omega_{2s}^1\sin 2\alpha + \Delta\Omega_{2s}^2\sin 4\alpha + 4A_2L_2^2(\cot I_0/\sin I_0)\Big\{\kappa_2\alpha\cos 2\alpha + e\Big[2\sin\alpha_P\cos 2\alpha - 3\sin(\alpha-\alpha_P) - \tfrac{1}{2}\sin(\alpha+\alpha_P) + \tfrac{7}{6}\sin(3\alpha-\alpha_P)\Big]\Big\} \tag{93}$$

and the J_2^2 coefficients are as follows:

$$\vartheta_{2s} = -\tfrac{4}{3}A_2^2L_2^2(\cot I_0/\sin I_0)\Big[3L_2^0 + 2\big(8-9\cot^2 I_0\big)L_2^2\Big]$$

$$\Delta\Omega_{2s}^1 = 2A_2^2(\cot I_0/\sin I_0)\Big[L_2^0 + 4\big(2-\cot^2 I_0\big)L_2^2\Big]$$

$$\Delta\Omega_{2s}^2 = -\tfrac{2}{3}A_2^2\big(L_2^2\big)^2(\cot I_0/\sin I_0)\big(4+3\cot^2 I_0\big) \tag{94}$$

Again we can modify the J_2 first-order ascending node o_2 and the second-order o_{2s} solutions by combining with terms of $\vartheta_2\kappa_2\alpha$, $\kappa_2\alpha\cos 2\alpha$, and $\mathcal{O}(J_2e)$, which appear in Eq. (93) to derive

$$o_2 = \vartheta_2(1+\kappa_2)\alpha + \Delta\Omega_2^1\sin 2\gamma$$

$$o_{2s}(\alpha) = \vartheta_{2s}\alpha + o_{2e}(\alpha) + \Delta\Omega_{2s}^1\sin 2\alpha + \Delta\Omega_{2s}^2\sin 4\alpha$$

$$o_{2e}(\alpha) = -\tfrac{3}{4}A_2e\cos I_0\Big[6\sin(\alpha-\alpha_P) - \sin(\alpha+\alpha_P) - \tfrac{1}{3}\sin(3\alpha-\alpha_P)\Big] \tag{95}$$

The radial equation (84) is rearranged to

$$s_{2s}''/a + s_{2s}/a = \tfrac{1}{2}e^2[1+3\cos(2\alpha-2\alpha_P)] + \varrho_{2s} + c_2\cos 2\alpha + c_4\cos 4\alpha + 4A_2L_2^2\Big\{\kappa_2\alpha\sin 2\alpha + 2e\Big[\sin\alpha_P\sin 2\alpha - \tfrac{4}{3}\cos(3\alpha-\alpha_P)\Big]\Big\} \tag{96}$$

where

$$\varrho_{2s} = -2A_2^2\Big[\big(L_2^0\big)^2 + \tfrac{1}{9}\big(13-72\cot^2 I_0\big)\big(L_2^2\big)^2\Big]$$

$$c_2 = -\tfrac{8}{3}A_2^2L_2^2\big(3L_2^0 - 14L_2^2\cot^2 I_0\big)$$

$$c_4 = \tfrac{10}{3}A_2^2\big(L_2^2\big)^2 \tag{97}$$

Notice that there are no $\sin(\alpha\pm\alpha_P)$ or $\cos(\alpha\pm\alpha_P)$ terms in Eq. (96). By substituting the ι_{2s} and o_{2s} solutions into Eq. (87), and neglecting all terms except those containing $\alpha\pm\alpha_P$, we have

$$s_{2s}''/a + s_{2s}/a = 0$$

$$2s_{2s}/a + \epsilon_{2s}' = K_{21}\cos(\alpha-\alpha_P) + K_{22}\cos(\alpha+\alpha_P) \tag{98}$$

where

$$K_{21} = -4A_2e\big(2L_2^0 - 3L_2^2\cot^2 I_0\big) = e(\varrho_2 + 3\kappa_2)$$

$$K_{22} = -2A_2eL_2^2\big(1-\cot^2 I_0\big) \tag{99}$$

If we eliminate s_{2s}/a in Eqs. (98) and integrate, we have

$$-(2s_{2s}'/a) + \epsilon_{2s} = K_{21}\sin(\alpha-\alpha_P) + K_{22}\sin(\alpha+\alpha_P) + C \tag{100}$$

where C is a constant of integration. By differentiating the second of Eqs. (98) and adding to Eq. (100), we obtain

$$\epsilon_{2s}'' + \epsilon_{2s} = C \tag{101}$$

Therefore, the first of Eqs. (98) and (101) claim that the $K_{21}\cos(\alpha-\alpha_P) + K_{22}\cos(\alpha+\alpha_P)$ term can be considered to be the combination of the general solutions of two oscillators in s_{2s} and ϵ_{2s}. The integration constant C can be chosen so that $\epsilon_{2s}=0$ when $\alpha=0$. Before conveniently allocating these terms to the s_{2s} and ϵ_{2s} equations, we solve Eq. (96) first. The special solution of Eq. (96) is

$$s_{2s}/a = \tfrac{1}{2}e^2[1-\cos(2\alpha-2\alpha_P)] + \varrho_{2s} + \Delta r_{2s}^1\cos 2\alpha + \Delta r_{2s}^2\cos 4\alpha - \tfrac{4}{3}A_2L_2^2\{\kappa_2\sin 2\alpha + e[2\sin\alpha_P\sin 2\alpha - \cos(3\alpha-\alpha_P)]\} \tag{102}$$

where two coefficients for the short-periodic term are given by

$$\Delta r_{2s}^1 = -c_2/3, \qquad \Delta r_{2s}^2 = -c_4/15 \tag{103}$$

Note that the first term in Eq. (102) is the second-order correction term for the original postepicycle equations. We shall allocate general solutions in Eq. (98) such that

$$2s_{2s}/a = -\tfrac{8}{3}A_2eL_2^2\cos(\alpha+\alpha_P)$$

$$\epsilon_{2s}' = (\varrho_2 + 3\kappa_2)e\cos(\alpha-\alpha_P) + \tfrac{2}{3}A_2eL_2^2\big(1+3\cot^2 I_0\big)\cos(\alpha+\alpha_P) \tag{104}$$

For the s_{2s} solution in Eq. (104), this term is necessary to complete the conversion from $\cos 2\alpha$ to $\cos 2\gamma$ in the original J_2 first-order r solution (as we did for the ι_{2s} and o_{2s} solutions).

Combining the general solution (104) and the last term of the s_{2s} solution (102) with the J_2 first-order solution, we obtain the modified s_2 and s_{2s} solution, which is given by

$$s_2 = -ae\cos(\alpha-\alpha_P) + a\varrho_2 + \Delta r_2^1\cos 2\gamma$$

$$s_{2s} = \tfrac{1}{2}ae^2[1-\cos(2\alpha-2\alpha_P)] + a\varrho_{2s} + \Delta r_{2s}^1\cos 2\alpha + \Delta r_{2s}^2\cos 4\alpha \tag{105}$$

Substituting all of the first- and second-order solutions we have derived into Eq. (87) and integrating it, we complete the solution for the last parameter ϵ_{2s}, which is given by

$$\epsilon_{2s} = \epsilon_{2s}(\alpha) - \epsilon_{2s}(0)$$

$$\epsilon_{2s}(\alpha) = \tfrac{5}{4}e^2 \sin(2\alpha - 2\alpha_P) + \kappa_{2s}\alpha + \Delta\lambda_{2s}^1 \sin 2\alpha + \Delta\lambda_{2s}^2 \sin 4\alpha$$
$$+ (\varrho_2 + 3\kappa_2)e\sin(\alpha - \alpha_P) + \tfrac{2}{3}A_2L_2^2\Big\{\kappa_2\big(1 - 6\cot^2 I_0\big)$$
$$\times \alpha\cos 2\alpha - e\Big[2\big(1 - 6\cot^2 I_0\big)\cos 2\alpha - \big(1 + 3\cot^2 I_0\big)$$
$$\times \sin(\alpha + \alpha_P) - \big(1 - 7\cot^2 I_0\big)\sin(3\alpha - \alpha_P)\Big]\Big\} \tag{106}$$

where

$$\kappa_{2s} = A_2^2\Big[9\big(L_2^0\big)^2 - 8L_2^0L_2^2\cot^2 I_0$$
$$- \tfrac{4}{3}\big(7 - 61\cot^2 I_0 + 6\cot^4 I_0\big)\big(L_2^2\big)^2\Big]$$

$$\Delta\lambda_{2s}^1 = -A_2^2L_2^2\Big[\big(1 + 2\cot^2 I_0\big)L_2^0 + \tfrac{4}{9}\cot^2 I_0\big(47 - 18\cot^2 I_0\big)L_2^2\Big]$$

$$\Delta\lambda_{2s}^2 = -\tfrac{1}{9}A_2^2\big(L_2^2\big)^2\big(2 - 33\cot^2 I_0 - 18\cot^4 I_0\big) \tag{107}$$

Again, the first term of $\epsilon_{2s}(\alpha)$ in Eq. (106) describes the postepicycle correction to the orbit. Like other solutions, adding $\kappa_2\alpha\cos 2\alpha$ and $\mathcal{O}(J_2e)$ terms to the first-order solution, we can modify the ϵ_2 as well as ϵ_{2s} solution such that

$$\epsilon_2 = 2e[\sin(\alpha - \alpha_P) + \sin\alpha_P] + \kappa_2\alpha + \lambda_2^1 \sin 2\gamma$$

$$\epsilon_{2s}(\alpha) = \tfrac{5}{4}e^2\sin(2\alpha - 2\alpha_P) + \kappa_{2s}\alpha + \epsilon_{2e}(\alpha)$$
$$+ \Delta\lambda_{2s}^1 \sin 2\alpha + \Delta\lambda_{2s}^2 \sin 4\alpha$$

$$\epsilon_{2e}(\alpha) = (\varrho_2 + 3\kappa_2)e\sin(\alpha - \alpha_P) - \tfrac{1}{4}A_2e\Big[\big(3 - 5\sin^2 I_0\big)$$
$$\times \sin(\alpha + \alpha_P) + \cos^2 I_0 \sin(3\alpha - \alpha_P)\Big] \tag{108}$$

We provide explicit expressions for these $\mathcal{O}(J_2^2)$ postepicycle coefficients in Appendix B.

VII. Results

In this section we shall demonstrate the accuracy over time of the model we have presented and show that we can accurately estimate osculating orbital elements over a period of around 5000 orbits for a typical low-Earth-orbiting satellite.

The simulation we shall present compares the epicycle description of the orbit using terms up to and including J_4. The analytic description is then compared with the output of a sophisticated orbit modeler based on a second-order Bulirsch–Stoer integrator (see Ref. 13). This integration method provides high levels of orbital accuracy, as well as very short integration times.

We define our starting conditions in terms of the position and velocity of the satellite at some initial epoch $t = 0$. The computation of the epicycle parameters from the position and velocity is very straightforward. The osculating orbital plane is defined as the plane containing the position and velocity vector, and once this plane has been defined we can immediately determine the inclination I_0 and right ascension of the ascending node Ω_0. These angles just define the orbital plane with respect to the inertial coordinate system. From the satellite's position and velocity we can compute the orbital energy, keeping in all of the relevant terms in the geopotential (in our case up to and including J_4). The semimajor axis a is then determined directly from this energy.

All that remains for the epicycle description of the orbit are the epicycle amplitude A and epicycle phase at perigee α_P. These appeared as constants of integration in our radial equation. Because everything else is known, through the initial conditions of radial position of the satellite r_0 and the radial velocity r_0', we can determine the remaining unknowns. This completes the set of epicycle parameters that we require.

We have chosen a low Earth orbit with a semimajor axis of 7223 km, for example, 845-km orbital altitude. The numerically integrated orbit and the analytic orbit are both propagated for 360 days (which corresponds to more than 5000 orbits) to determine the magnitude of errors.

In Fig. 1, we have plotted the peak positional error in terms of along, cross, and radial directions during 360 days propagation as a function of inclination angle. We have chosen null epicycle amplitude, for example, $A = 0$, to obtain the result. The error plotted is normalized by the semimajor axis. It appears to give worse along-track error as the inclination angle goes smaller. This result shows, however, the dominant along-track peak errors are $\mathcal{O}(10^{-4})$ for all inclinations. If we consider 5000 orbit propagation time, for example, epicycle phase α reaches more than 3×10^4 rad, this order of

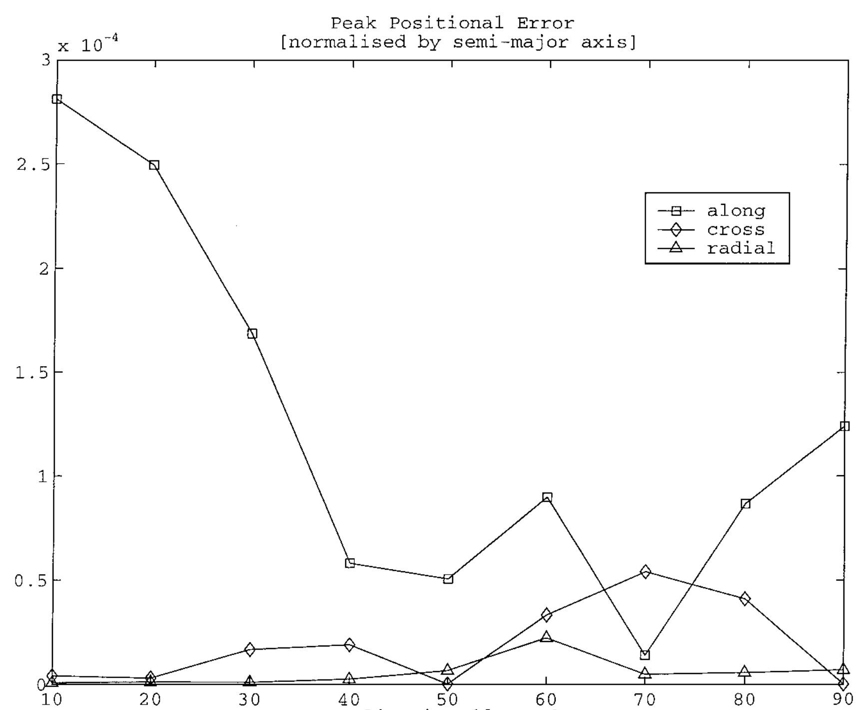

Fig. 1 Peak positional error in along, cross track, and radial direction during 5000 orbits propagation (normalized by semimajor axis) as a function of inclination angle (degree); zero epicycle amplitude is chosen.

along-track error indicates that the error in semimajor axis, $\Delta a/a$, is $\mathcal{O}(10^{-8}) \sim \mathcal{O}(10^{-9})$.

The peak positional error as a function of (normalized) epicycle amplitude is presented in Fig. 2, where the inclination angle is fixed to be 98 deg to obtain the result. The plot shows that it seems to have a minimum at $A/a(\equiv e) = 1.5 \times 10^{-3}$ and that the error is getting significant as the epicycle amplitude becomes larger. If we do the same simulation by altering the inclination 10 deg, the peak along-track error reaches 2.6×10^{-3} at $e = 5 \times 10^{-3}$ corresponding to the error in semimajor axis of $\Delta a/a \approx 5.5 \times 10^{-8}$. We expect, however, such results because we have imposed the assumption of $\mathcal{O}(e) \approx \mathcal{O}(J_2)$ to derive epicycle formulas and how the ordering scheme of perturbation terms are arranged. If we set $e = 10^{-2}$, one order larger than J_2, then no matter what the inclination is, we get the result that shows the error in semimajor axis $\Delta a/a$ can go up to $\mathcal{O}(10^{-7})$.

In Fig. 3, the history of along-track propagation errors with respect to time is presented. We have used the values of 10^{-3} for e

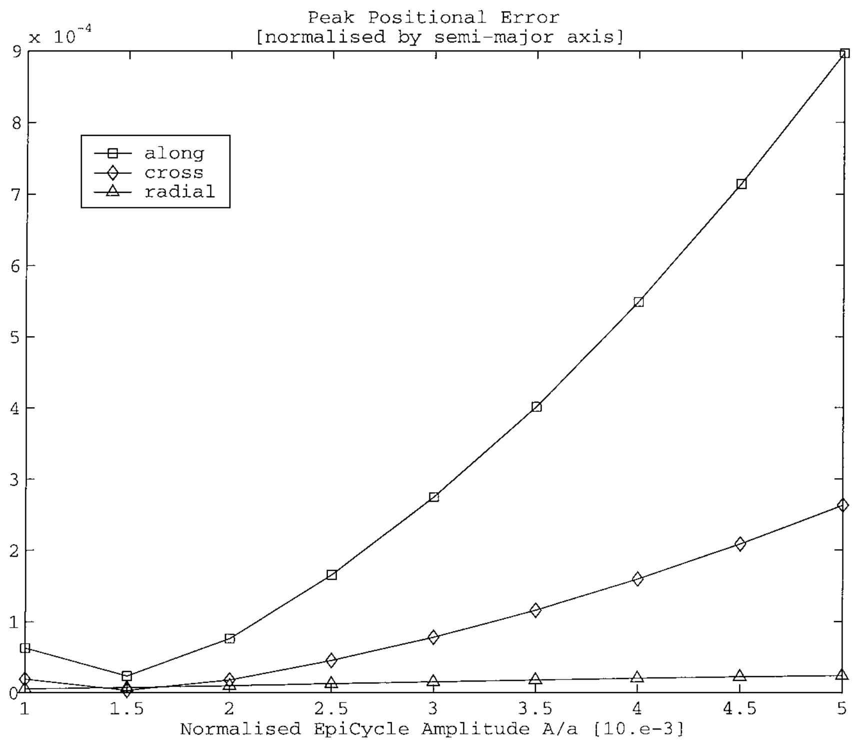

Fig. 2 Peak positional error in along, cross track, and radial direction during 5000 orbits propagation (normalized by semimajor axis) as a function of normalized epicycle amplitude *A*/*a* (or eccentricity); inclination angle is 98 deg.

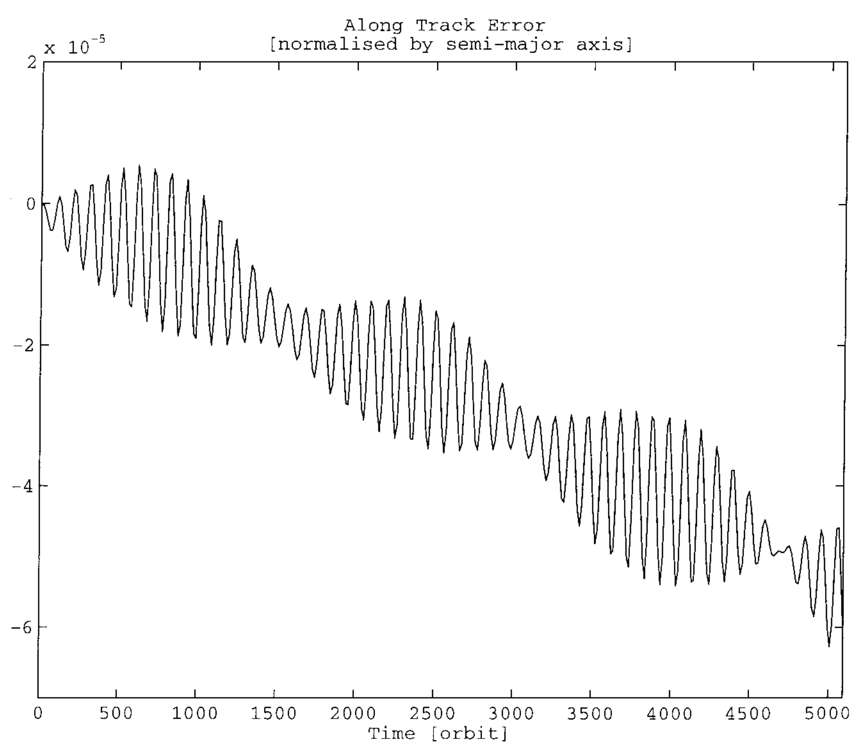

Fig. 3 Error in along track direction (normalized by semimajor axis) as a function of propagation time (orbit); normalized epicycle amplitude *A*/*a* (or eccentricity) is 10^{-3} and inclination angle is 98 deg.

and 98 deg for the inclination. We can see the secular drift error in the result; however, this magnitude of secular error after 5000 orbits propagation can be due to the error in semimajor axis $\Delta a/a$ of $\mathcal{O}(10^{-9})$. Because we are not modeling the perturbation terms of $J_2 e^2$, J_2^3, and $J_2 J_4$, which are all $\mathcal{O}(10^{-9})$, the result we have obtained in Fig. 3 is quite satisfactory.

VIII. Conclusion

We have presented the solution for the perturbed motion of a satellite in a near-circular orbit about an oblate planet. Our formulation differs from that used in classical papers in that we avoid the definition of a mean orbital plane and mean elements. We do define a mean orbital radius, about which we expand, but this is easily defined in terms of the orbital energy, which is conserved by the motion of the satellite. The formulation we have presented is valid for orbital eccentricities of comparable order to J_2 or smaller and leads to a very simple geometric interpretation. The effects due to short-period, long-period, and secular variations are all separated in the perturbed equations, and the general motion can be described in compact form.

The position of the satellite is described by two osculating elements that define the instantaneous orbital plane (the plane containing the position and velocity vectors) and polar coordinates (r, λ) on the orbital plane, where λ is the argument of latitude measured from the initial ascending node. This representation is redundant, because it requires four coordinates to locate the satellite, but greatly simplifies the description of the motion. The use of the argument of latitude allows time to be easily described in terms of our epicycle phase, rather than needing to solve Kepler's equation, and combines the variation in the mean motion and the argument of perigee into a single angular position for the satellite.

We have compared the predictions of these analytic equations with numerical simulations including J_2, J_3, and J_4 terms. We have shown that the errors in the semimajor axis of the satellite is of order $\mathcal{O}(10^{-9}) \sim \mathcal{O}(10^{-8})$, which is to be expected when neglecting terms such as J_2^3 or $J_2 J_4$. We have shown that the growth in position errors over a period of 5000 orbits is still acceptably small. This makes the formulation suitable for onboard orbit determination on a satellite, where the epicycle parameters can be estimated each day.

Appendix A: Epicycle Coefficients

We provide here the nonzero coefficients for the dominant terms in the epicycle equations J_2, J_3, and J_4.

The J_2 coefficients for the secular terms are

$$\varrho_2 = -\tfrac{1}{4} J_2 (R/a)^2 \left(2 - 3\sin^2 I_0\right), \qquad \vartheta_2 = -\tfrac{3}{2} J_2 (R/a)^2 \cos I_0$$

$$\kappa_2 = \tfrac{3}{4} J_2 (R/a)^2 \left(4 - 5\sin^2 I_0\right) \tag{A1}$$

and for short-periodic terms are

$$\Delta r_2^1 = \tfrac{1}{4} J_2 (R/a)^2 a \sin^2 I_0, \qquad \Delta I_2^1 = -\tfrac{3}{8} J_2 (R/a)^2 \sin 2I_0$$

$$\Delta \Omega_2^1 = \tfrac{3}{4} J_2 (R/a)^2 \cos I_0$$

$$\Delta \lambda_2^1 = -\tfrac{1}{8} J_2 (R/a)^2 \left(6 - 7\sin^2 I_0\right) \tag{A2}$$

The J_3 coefficients for the long-periodic terms using only J_2 in κ_2 are

$$\chi_3 = (J_3 / 2J_2)(R/a) \sin I_0 \tag{A3}$$

and short-periodic terms are

$$\Delta I_3^0 = -\tfrac{3}{8} J_3 (R/a)^3 \cos I_0 \left(4 - 5\sin^2 I_0\right)$$

$$\Delta \Omega_3^0 = \tfrac{3}{8} J_3 (R/a)^3 \cot I_0 \left(4 - 15\sin^2 I_0\right)$$

$$\Delta \lambda_3^0 = \tfrac{3}{8} J_3 (R/a)^3 \sin I_0 \Big[\left(4 - 5\sin^2 I_0\right) - \left(4 - 15\sin^2 I_0\right)\cot^2 I_0\Big] \tag{A4}$$

and

$$\Delta r_3^1 = \tfrac{5}{32} J_3 (R/a)^3 a \sin^3 I_0, \qquad \Delta I_3^1 = \tfrac{5}{8} J_3 (R/a)^3 \cos I_0 \sin^2 I_0$$

$$\Delta \Omega_3^1 = \tfrac{5}{8} J_3 (R/a)^3 \cot I_0 \sin^2 I_0$$

$$\Delta \lambda_3^1 = -\tfrac{5}{48} J_3 (R/a)^3 \sin I_0 \left(6 - 7\sin^2 I_0\right) \tag{A5}$$

The J_4 coefficients for the secular terms are

$$\varrho_4 = \tfrac{9}{64} J_4 (R/a)^4 \left(8 - 40\sin^2 I_0 + 35\sin^4 I_0\right)$$

$$\vartheta_4 = \tfrac{15}{16} J_4 (R/a)^4 \cos I_0 \left(4 - 7\sin^2 I_0\right)$$

$$\kappa_4 = -\tfrac{3}{64} J_4 (R/a)^4 \left(136 - 500\sin^2 I_0 + 385\sin^4 I_0\right) \tag{A6}$$

and for short-periodic terms are

$$\Delta r_4^1 = -\tfrac{5}{16} J_4 (R/a)^4 a \sin^2 I_0 \left(6 - 7\sin^2 I_0\right)$$

$$\Delta I_4^1 = \tfrac{5}{32} J_4 (R/a)^4 \sin 2I_0 \left(6 - 7\sin^2 I_0\right)$$

$$\Delta \Omega_4^1 = -\tfrac{5}{8} J_4 (R/a)^4 \cos I_0 \left(3 - 7\sin^2 I_0\right)$$

$$\Delta \lambda_4^1 = \tfrac{5}{32} J_4 (R/a)^4 \left(12 - 34\sin^2 I_0 + 21\sin^4 I_0\right) \tag{A7}$$

$$\Delta r_4^2 = -\tfrac{7}{64} J_4 (R/a)^4 a \sin^4 I_0$$

$$\Delta I_4^2 = \tfrac{35}{128} J_4 (R/a)^4 \sin 2I_0 \sin^2 I_0$$

$$\Delta \Omega_4^2 = -\tfrac{35}{64} J_4 (R/a)^4 \cos I_0 \sin^2 I_0$$

$$\Delta \lambda_4^2 = \tfrac{7}{256} J_4 (R/a)^4 \sin^2 I_0 \left(20 - 23\sin^2 I_0\right) \tag{A8}$$

We note that the expression for the drift in epicycle phase κ_4 in Eqs. (A6) does not appear to agree with other published formulations of secular perturbation in argument of perigee, for example, Ref. 1. The reason for this apparent disagreement arises from the different definitions of mean orbital radius that appear in the literature. In Eq. (16), we have shown our mean motion n is related to $\bar{n}_0$, which is the averaged mean motion around the orbit. The J_4 contribution to this relation is, therefore, $n = \bar{n}_0(1 - \varrho_4)$. If we reformulate our $\kappa_4 n$ term by using $\bar{n}_0$, we find

$$\kappa_4 n = (\kappa_4 - \varrho_4)\bar{n}_0 = -\tfrac{15}{32} J_4 (R/a)^4 \left(16 - 62\sin^2 I_0 + 49\sin^4 I_0\right)\bar{n}_0 \tag{A9}$$

which agrees with Kozai[1] if terms of $\mathcal{O}(J_4 e^2)$ are ignored.

Appendix B: Postepicycle Coefficients

We provide here the nonzero J_2^2 coefficients in the postepicycle equations.

Secular terms:

$$\varrho_{2s} = -\tfrac{1}{32} J_2^2 (R/a)^4 \left(16 + 24\sin^2 I_0 - 49\sin^4 I_0\right)$$

$$\vartheta_{2s} = \tfrac{3}{8} J_2^2 (R/a)^4 \cos I_0 \left(11 - 20\sin^2 I_0\right)$$

$$\kappa_{2s} = \tfrac{3}{16} J_2^2 (R/a)^4 \left(14 + 17\sin^2 I_0 - 35\sin^4 I_0\right) \tag{B1}$$

J_2^2 2α-short-periodic terms:

$$\Delta r_{2s}^1 = -\tfrac{1}{4} J_2^2 (R/a)^4 a \sin^2 I_0 \left(9 - 10\sin^2 I_0\right)$$

$$\Delta I_{2s}^1 = \tfrac{3}{32} J_2^2 (R/a)^4 \sin 2I_0 \left(11 - 15\sin^2 I_0\right)$$

$$\Delta \Omega_{2s}^1 = -\tfrac{3}{16} J_2^2 (R/a)^4 \cos I_0 \left(8 - 21\sin^2 I_0\right)$$

$$\Delta \lambda_{2s}^1 = \tfrac{1}{32} J_2^2 (R/a)^4 \left(48 - 190\sin^2 I_0 + 139\sin^4 I_0\right) \tag{B2}$$

J_2^2 4α-short-periodic terms:

$$\Delta r_{2s}^2 = -\tfrac{1}{32} J_2^2 (R/a)^4 a \sin^4 I_0$$

$$\Delta I_{2s}^2 = \tfrac{3}{128} J_2^2 (R/a)^4 \sin 2I_0 \left(3 + 5\sin^2 I_0\right)$$

$$\Delta \Omega_{2s}^2 = -\tfrac{3}{32} J_2^2 (R/a)^4 \cos I_0 \left(3 + \sin^2 I_0\right)$$

$$\Delta \lambda_{2s}^2 = \tfrac{1}{64} J_2^2 (R/a)^4 \left(18 - 3\sin^2 I_0 - 17\sin^4 I_0\right) \tag{B3}$$

Note again J_2 second-order secular coefficients, may differ from other published results. This is, however, due to the different method of averaging the orbital elements. Considering the ascending node coefficient ϑ_{2s} in Eqs. (B1), we introduce the notation

$$\bar{a}_0 = a(1 - 2\varrho_2), \qquad i_0 = I_0 - \tfrac{3}{8} J_2 (R/a)^2 \sin 2I_0 \tag{B4}$$

where $\bar{a}_0$ is the averaged semimajor axis used by Kozai.[1] This he relates to an averaged mean motion $\bar{n}_0 = n(1 + 3\varrho_2)$, which satisfies $\bar{a}_0^3 \bar{n}_0^2 = \mu$. The second of Eqs. (B4) represents the relationship between Kozai's definition of initial inclination and ours. If we combine the first- and second-order expressions for the secular evolution in ascending node, expressed in terms of these osculating quantities, we find

$$\dot{\Omega}/n = -\tfrac{3}{2} J_2 (R/\bar{a}_0)^2 \cos i_0 - \tfrac{3}{8} J_2^2 (R/\bar{a}_0)^4 \cos i_0 \left(9 - 10\sin^2 i_0\right) \tag{B5}$$

which agrees with Kozai[1] if terms of $\mathcal{O}(J_2 e^2,\, J_2^2 e^2)$ are ignored.

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